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TOPOLOGICAL ANALYSIS OF NATURAL SYSTEMS WITH QUADRATIC INTEGRALS*

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A method for the qualitative investigation of natural systems is considered, which enables integrals that are quadratic in velocities to be used based on a description of the surfaces of integral levels in the phase space. The concept of the normalized quadratic integral and the connection between its presence and the separation of positional variables is established. The method of topological analysis, proposed in /1/ for systems with linear integrals, is extended to problems containing quadratic integrals. Surfaces of the integral levels, their bifurcations, and the regions of possible motion for given values of the integrals are described. As an example of their application, the motion of a solid in a potential field is considered.

1. The quadratic integrals. Their normalizability. All objects are assumed to be smooth. Let M be the n-dimensional configuration manifold having the metric $\langle\cdot,\cdot\rangle$, let x be its point, and let TM be the tangent lamination and T_xM the lamination over the point x. For the laminar linear operator $\Gamma: TM \to TM$ the symbol Γ_x denotes its contraction on T_xM. We will denote the vector field on M by the same symbol as its arbitrary vector; the distinction is clear from the context. The symbol of form $\mathbf{v}_x$ denotes either a separate element T_xM or a vector of field $\mathbf{v}$ applied at the point x.

Let ∇ be the operator of covariant differentiation of the function V on M, the potential. The trajectory of the system is defined by Newton's equation

$$\nabla_{\mathbf{v}}\mathbf{v} = -\operatorname{grad} V$$

where $\mathbf{v}_x$ is the velocity vector at the point $x \in M$, and the energy integral

$$H(\mathbf{v}_x) = {}^1/_2\,|\mathbf{v}_x|^2 + V(x)$$

exists.

Consider the function $G: TM \to R$ which is quadratic in the velocities

$$G(\mathbf{v}_x) = {}^1/_2\,\langle\Gamma\mathbf{v}_x,\ \mathbf{v}_x\rangle + W(x) \tag{1.1}$$

where $\Gamma: TM \to TM$ is a symmetric laminar linear operator and W is a function on M.

The necessary and sufficient condition for G to be the first integral is expressed by the equations /3/

$$\langle\nabla_{\mathbf{v}}\Gamma\mathbf{u},\mathbf{w}\rangle + \langle\nabla_{\mathbf{u}}\Gamma\mathbf{w},\mathbf{v}\rangle + \langle\nabla_{\mathbf{w}}\Gamma\mathbf{v},\mathbf{u}\rangle = 0, \qquad \Gamma\operatorname{grad} V = \operatorname{grad} W \tag{1.2}$$

It can be shown that the lack of a term that is linear in the velocites does not lead to any loss of generality.

Definition. We shall call normal those coordinates of the quadratic integral (1.1) on M whose basis vectors at every point are eigenvectors of the operator Γ_x: $T_xM \to T_xM$, and we shall call (1.1) normalizable, if normal coordinates exist for it, and non-normalizable otherwise.

The existence of the normalizable quadratic integral is closely connected with the separation of positional variables.** (**Orekhov V.I. On the separation of variables in natural systems with quadratic integrals. Moscow, 1979. Manuscript deposited in VINITI 29.02. 79, No.720.) The problem admits of a normalizble quadratic integral with non-coinciding eigenvalues of the quadratic part, if and only if its normal coordinates are Stäckel coordinates. Owing to the separation of these variables into Hamilton-Jacobi equations, the problem in this case is fully integrable /4/. If a non-normalizable quadratic integral exists and the system is non-degenerate, and the separation of positional variables is impossible.

Let us determine the conditions for integral (1.1) to be normalizable. The operator Γ_x has n different real eigendirections at each point $x \in M$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be smooth vector fields such that at points of the general position the vectors $\mathbf{u}_{ix}$ constitute the basis space T_xM consisting of the eigenvectors Γ_x. Let

$$[\mathbf{u}_i, \mathbf{u}_j] = \sum_{k=1}^{n} c_{ij}{}^k\mathbf{u}_k, \qquad i, j = 1, \ldots, n$$

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The integral will be normalizable, if functions $\vartheta_1, \ldots, \vartheta_n$ can be found on M such that the fields $\vartheta_i \mathbf{u}_i$ become basis fields for some coordinates, i.e. will be pairwise commute

$$0 = [\vartheta_i \mathbf{u}_i, \vartheta_j \mathbf{u}_j] = \vartheta_i \mathbf{u}_i(\vartheta_j) - \vartheta_j \mathbf{u}_j(\vartheta_i) + \vartheta_i \vartheta_j [\mathbf{u}_i, \mathbf{u}_j]$$

which is equivalent to the relations

$$c_{ij}{}^k = 0, \quad i, j \neq k \tag{1.3}$$

$$\mathbf{u}_i (\ln \vartheta_j) = c_{ji}{}^i, \quad i \neq j \tag{1.4}$$

For each j (1.4) yields a system of $(n-1)$ linear differential equations in $\ln \vartheta_j$. The conditions of its complete integration in view of (1.3) have the form

$$\mathbf{u}_i (c_{kj}{}^k) - \mathbf{u}_j (c_{ki}{}^k) = c_{ij}{}^i c_{ki}{}^k + c_{ji}{}^j c_{kj}{}^k$$

The equations together with (1.3) define the normalization condition.

Note that any quadratic integral is normalizable in a system with two degrees of freedom. Indeed, when $n = 2$ conditions (1.3) is eliminated, and from (1.4) two equations are obtained which are integrated independently of one another.

2. Levels of quadratic integrals. Characteristic functions. Consider the integral (1.1). Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the basis vector fields introduced in Sect.1, with consist of eigenvectors of the operator Γ, and $\lambda_1(x), \ldots, \lambda_n(x)$ their corresponding eigenvalues. We assume that everywhere on M we have $\lambda_1 > \ldots > \lambda_n$. Because of the orthogonality of the fields $\mathbf{u}_i$, from (1.2) it follows that

$$\begin{aligned} &\lambda_i \mathbf{u}_i (V) = \mathbf{u}_i (W), \quad \mathbf{u}_i (\lambda_i) = 0 \\ &|\mathbf{u}_i|^2 \mathbf{u}_j (\lambda_i) = (\lambda_i - \lambda_j)(\mathbf{u}_j (|\mathbf{u}_i|^2) + 2\langle [\mathbf{u}_i, \mathbf{u}_j], \mathbf{u}_i \rangle) \\ &i \neq j \end{aligned} \tag{2.1}$$

Let us construct the integral representation of $I = G \times H : TM \to R^2$, whose levels are the integral sets

$$I_{gh} = I^{-1}(g, h) = \{\mathbf{v}_x \in TM : G(\mathbf{v}_x) = g, \ H(\mathbf{v}_x) = h\}$$

For regular values of I, which are points of general position on the plane $R^2 = \{(g, h)\}$, and the sets I_{gh} are smooth manifolds that retain their type for small variations of g, and h. The critical values of I form bifurcation curves in the $\{(g, h)\}$ plane the integrals of the manifold I_{gh} passing across them undergo a reconstruction.

In the description of the structure of integral manifolds and their bifurcations a key part is played by the n functions of positional variables which we shall call characteristic. These are the functions

$$\Phi_i = \lambda_i (h - V) + W - g \tag{2.2}$$

which parametrically depend on the constants g and h. We denote the characteristic functions which correspond to fixed values of these constants by $\Phi_i |_{gh}$. By virtue of (2.1) we have $\mathbf{u}_i (\Phi_i) = 0$, i.e. the surface levels of the functions $\Phi_i |_{gh}$ are invariant to the field $\mathbf{u}_i$ and their cirtical points are degenerate.

The following theorem defines the characteristic functions for the topological analysis.

Theorem. The critical points $\mathbf{v}_x \in TM$ of the integral representation are determined by one of the following n conditions $(i = 1, \ldots, n)$:

$$\mathbf{v}_x \parallel \mathbf{u}_{ix}, \quad d\Phi_i |_{gh} = 0, \quad g = G(\mathbf{v}_x), \quad h = H(\mathbf{v}_x)$$

Proof. The differentials $dG, dH : T_{\mathbf{v}} TM \to R$ are proportional at the critical point $\mathbf{v}_x$. We separate in every tangent space $T_{\mathbf{v}} TM$ a vertical subspace $T_{\mathbf{v}} T_x M$. The proportionality of dG and dH on $T_{\mathbf{v}} T_x M$ means the proportionality of the partial derivatives of the quadratic forms $\langle \Gamma \mathbf{v}_x, \mathbf{v}_x \rangle$ and $|\mathbf{v}_x|^2$ with respect to the velocity variables, which gives $\mathbf{v}_x \parallel \mathbf{u}_{ix}$, and the proportionality coefficient is equal to $\lambda_i(x)$.

Thus the cirtical points $\mathbf{v}_x$ are contained in one of the n laminations over M with a one-dimensional layer generated by $\mathbf{u}_i$. The critical points are separated from these laminations by the condition of proportionality of dG and dH with the same coefficient on any subspace of the $T_{\mathbf{v}} TM$ space, which is transverse to the vertical $T_{\mathbf{v}} T_x M$. Such a subspace may be considered as the tangent space to the set $\{\mathbf{v}_x = \vartheta \mathbf{u}_{ix}\}$, where $\vartheta(x)$ is some functions on M. Hence for the critical points

$$dG(\vartheta \mathbf{u}_{ix}) = \lambda_i(x)\, dH(\vartheta \mathbf{u}_{ix}) \tag{2.3}$$

Since $\vartheta \mathbf{u}_{ix}$ are eigenvectors for the quadratic part of integral G, we have

$$G(\vartheta \mathbf{u}_{ix}) - W = \lambda_i(x)(H(\vartheta \mathbf{u}_{ix}) - V)$$

After differentiation of this identity it will be seen that condition (2.3) taking the equation $H(\vartheta \mathbf{u}_{ix}) = h$ into account, is equivalent to the condition

$$0 = d\lambda_i (h - V) + \lambda_i d(h - V) + d(W - g) = d\Phi_i |_{gh}$$

We shall denote the projection of the set I_{gh} onto the configuration manifold by M_{gh}, and call it the region of possible motions. It is the set of points through which the trajectories of motion may pass for fixed values of the integrals $G = g$, $H = h$. The section of I_{gh} by the tangent space T_xM represents a set of velocity vectors of possible motions passing through the point $x \in M$.

Each layer $T_xM \cap I_{gh}$ is formed by the intersection of second-order surfaces in T_xM

$$T_xM \cap I_{gh} = \{\mathbf{v}_x: \langle \Gamma \mathbf{v}_x, \mathbf{v}_x \rangle = 2(g - W)\} \cap \{\mathbf{v}_x: |\mathbf{v}_x|^2 = 2(h - V)\}$$

This intersection is non-empty when the condition

$$\lambda_n |\mathbf{v}_x|^2 \leqslant \langle \Gamma \mathbf{v}_x, \mathbf{v}_x \rangle \leqslant \lambda_1 |\mathbf{v}_x|^2$$

is satisfied, i.e. when $\lambda_n (h - V) \leqslant (g - W) \leqslant \lambda_1 (h - V)$, which by definition of the characteristic function is equivalent to the condition $\Phi_n |_{gh} \leqslant 0 \leqslant \Phi_1 |_{gh}$.

Thus the regions of possible motion are

$$M_{gh} = \{\Phi_1 |_{gh} \geqslant 0\} \cap \{\Phi_n |_{gh} \leqslant 0\}$$

The edges of this curvilinear polyhedron coincide with the intersections of the surfaces of level $\{V = h\}$ and $\{W = g\}$ and all surfaces $\{\Phi_i |_{gh} = 0\}$ pass through it. Indeed, since $\lambda_1 \neq \lambda_n$ the equations $\Phi_1 = \Phi_n = 0$ are equivalent to the conditions $V - h = W - g = 0$, and hence $\Phi_i = 0$, $i = 1, \ldots, n$.

At the critical points Φ_1 and Φ_n, as follows from (2.3), a reconstruction of the regions M_{gh} occurs. Bifurcation of the integral equations I_{gh} over the critical points of the remaining characteristic functions are revealed by the reconstruction of the edges of M_{gh}: if $d\Phi_i = 0$ and $V = h$, then from (2.2) it follows that $-\lambda_i dV + dW = 0$, i.e. dV and dW are proportional.

The cuts $T_xM \cap I_{gh}$ maintain their topological type under condition $\lambda_{j+1} |\mathbf{v}_x|^2 < \langle \Gamma \mathbf{v}_x, \mathbf{v}_x \rangle < \lambda_j |\mathbf{v}_x|^2$, which is equivalent to $\Phi_{j+1} |_{gh} < 0 < \Phi_j |_{gh}$, $j = 1, \ldots, n - 1$, and are reconstructed when $\lambda_i |\mathbf{v}_x|^2 = \langle \Gamma \mathbf{v}_x, \mathbf{v}_x \rangle$, i.e. over points $\{\Phi_i |_{gh} = 0\}$. Over the edges $\{\Phi_1 = 0\}$ and $\{\Phi_n = 0\}$ the cut degenerates into a pair of vectors $\mathbf{v} \parallel \mathbf{u}_{1,n}$ tangent to the edges.

The set of critical points of the characteristic functions Φ_i which are invariant relative to the fields $\mathbf{u}_i$ carry in themselves the trajectories of the problem that are tangent to these fields.

Theorem. The integral curve of the field $\mathbf{u}_i$ is the trajectory of the field, if and only if it passes through the critical points of the function Φ_i.

Note that this trajectory lies on the surface $\{\Phi_i = 0\}$.

Corollary. If the set of critical points of the zero level of the function Φ_i has a one-dimensional component, that component is the trajectory of steady motion with velocity

$$\mathbf{v} = \pm \sqrt{2(h - V)}\, |\mathbf{u}_i|^{-1} \mathbf{u}_i$$

of periodic, limit, or libration motion depending on the character of its intersection with Hill's surface $\{V = h\}$.

Proof of the theorem. Let $\mathbf{v}_x = \vartheta \mathbf{u}_{ix}$, $\vartheta^2 = 2(h - V) |\mathbf{u}_i|^{-2}$. By the properties of covariant differentiation taking into account the orthogonality of $\mathbf{u}_i$, relations (2.1) and $[\mathbf{u}, \mathbf{w}] = \nabla_{\mathbf{u}} \mathbf{w} - \nabla_{\mathbf{w}} \mathbf{u}$ we obtain

$$\langle \nabla_{\mathbf{v}} \mathbf{v} + \operatorname{grad} V, \mathbf{u}_i \rangle = \vartheta^{-1} \mathbf{v}(h) = 0$$
$$\langle \nabla_{\mathbf{v}} \mathbf{v} + \operatorname{grad} V, \mathbf{u}_j \rangle = (\lambda_j - \lambda_i)^{-1} \mathbf{u}_j (\Phi_i), \quad i \neq j$$

i.e. we have $\nabla_{\mathbf{v}} \mathbf{v} = -\operatorname{grad} V$, if and only if $\mathbf{u}_j (\Phi_i) = 0$, $j = 1, \ldots, n$.

3. Example: a solid in a Goryachev field.

Consider the motion of a solid in a field with the potential /2/

$$V = -A\varphi(\alpha, \beta, \gamma) - B\varphi(\alpha', \beta', \gamma') - C\varphi(\alpha'', \beta'', \gamma'') \tag{3.1}$$

where $A > B > C$ are the principal moments of inertia, $\alpha, \ldots, \gamma''$ are the direction cosines of the principal axes of inertia relative to the stationary frame of reference $\mathbf{e}_{1,2,3}$ and φ is a quadratic form with constant coefficients. Such a potential approximates, for example, the gravitational action of an arbitrary distributed mass on a body fixed at the centre of gravity.

In this case the configuration manifold is $M \simeq SO(3)$, $n = 3$; the elements TM ca be identified with the angular velocity vectors in three-dimensional space.

Suppose the fixed frame of reference coincides with the principal axes of the form φ. Then

$$\varphi(\alpha, \beta, \gamma) = a_1\alpha^2 + a_2\beta^2 + a_3\gamma^2, \ldots$$

If all eigenvalues a_i are the same, then $V = \text{const}$; when only two of them are the same, a de Brun potential is obtained.

Consider the case of different eigenvalues $a_1 > a_2 > a_3$.

If we add to each a_i the same term, potential (3.1) in view of $\alpha^2 + \beta^2 + \gamma^2 = 1$ etc. becomes a constant. Below we will assume that $a_1 + a_2 + a_3 = 0$. Then

$$\varphi(\alpha, \beta, \gamma) + \varphi(\alpha', \beta', \gamma') + \varphi(\alpha'', \beta'', \gamma'') = 0 \tag{3.2}$$

According to /2/, besides the energy integral H the problem admits of another quadratic integral G; we have in standard notation

$$H = {}^1/_2\,(Ap^2 + Bq^2 + Cr^2) + V, \quad G = {}^1/_2\,(A^2p^2 + B^2q^2 + C^2r^2) + W$$
$$W = BC\varphi(\alpha, \beta, \gamma) + CA\varphi(\alpha', \beta', \gamma') + AB\varphi(\alpha'', \beta'', \gamma'')$$

The eigenvalues of the quadratic part of G are $\lambda_A = A,\ \lambda_B = B,\ \lambda_C = C$. As the eigenvectors we take $\mathbf{u}_A, \mathbf{u}_B, \mathbf{u}_C$ which are the unit vectors of the respective axes of inertia, which determine rotation around these axes at unit velocity. Since $[\mathbf{u}_A, \mathbf{u}_B] = \mathbf{u}_C$ etc., the normalization conditions are not satisfied and the integral G is not normalizable.

The characteristic functions, taking condition (3.2) into account, have the form

$$\Phi_A = (A - B)(A - C)\,\varphi(\alpha, \beta, \gamma) + Ah - g$$
$$(ABC,\ \alpha\alpha'\alpha'',\ \beta\beta'\beta'',\ \gamma\gamma'\gamma'')$$

and their ciritcal points coincide with the cirtical points $\varphi(\alpha, \beta, \gamma)$, $\varphi(\alpha', \beta', \gamma')$, $\varphi(\alpha'', \beta'', \gamma'')$ respectively. It is convenient to change from the constants g, h to the parameters l_1, l_2

$$l_1 = \frac{g - Ah}{(A-B)(A-C)}, \quad l_2 = \frac{g - Ch}{(C-A)(C-B)}, \quad \frac{g - Bh}{(B-C)(B-A)} = -(l_1 + l_2)$$

The conditions $\Phi_A \geqslant 0,\ \Phi_C \leqslant 0,\ \Phi_B = 0$ are then equivalent to the relations $\varphi(\alpha, \beta, \gamma) \geqslant l_1$, $\varphi(\alpha'', \beta'', \gamma'') \leqslant l_2$, $-\varphi(\alpha', \beta', \gamma') = (l_1 + l_2)$.

The regions where motion is possible are then

$$M_{gh} = \{\varphi(\alpha, \beta, \gamma) \geqslant l_1\} \cap \{\varphi(\alpha'', \beta'', \gamma'') \leqslant l_2\}$$

The reconstructions of curves of possible velocities occurs over the points $\{\varphi(\alpha', \beta', \gamma') = -(l_1 + l_2)\}$.

Bifurcation of the integral levels I_{gh} occurs when $l_1, l_2, -(l_1 + l_2)$ pass through the critical values of φ, i.e. when $l_1 = a_i,\ l_2 = a_i,\ l_1 + l_2 = -a_i,\ i = 1, 2, 3$. The positions of the solid when $\mathbf{u}_{A,B,C}$ are collinear to one of the vectors of the fixed frame of reference correspond to points of the characteristic functions, at which bifurcation occurs.

The rotations of the solid around its principal axis of inertia that and collinear with the vector of the fixed axis of reference $\mathbf{e}_i$, define the steady motions described in Sect.2. Let us consider the motion when $\mathbf{u}_A \parallel \mathbf{e}_1$: then $l_1 = a_1$. When the change of initial conditions of the quantities $(a_1 - l_1)$ is small, the quantity M_{gh} will be a small cylindrical neighbourhood of the steady-motion trajectory. Consequently the motion considered here is stable with respect to some of the variables. Likewise, the steady motion is stable with respect to some of the variables, when $\mathbf{u}_C \parallel \mathbf{e}_3$.

Let the eigenvalues of the form φ be connected by the condition $a_1 - a_2 = a_2 - a_3$. In this case, from the stipulation $a_1 + a_2 + a_3 = 0$ it follows that $a_2 = 0,\ a_1 = -a_3 = a > 0$ and

$$V = -a\,[A(\alpha^2 - \gamma^2) + B(\alpha'^2 - \gamma'^2) + C(\alpha''^2 - \gamma''^2)]$$
$$W = a\,[BC(\alpha^2 - \gamma^2) + CA(\alpha'^2 - \gamma'^2) + AB(\alpha''^2 - \gamma''^2)]$$

According to /2/ the problem with potential V, besides the integrals H and G, admits of a third quadratic integral

$$F = {}^1/_2\,[(A\alpha p + B\alpha' q + C\alpha'' r)^2 - (A\gamma p + B\gamma' q + C\gamma'' r)^2] + U$$
$$U = a\,[BC\beta^2 + CA\beta'^2 + AB\beta''^2)$$

We will introduce the parameters $g_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$, where $\mathbf{e}_i$ are vectors of the fixed frame of reference. The six variables $g_{ij} = g_{ji}$ may be considered as redundant local coordinates on $SO(3)$. In these variables $V = a\,(g_{11} - g_{33})$, $U = a\,(g_{11}g_{33} - g_{13}^2)$; and the eigenvalues of the quadratic part of the integral F are

$$\lambda_{1,3} = {}^1/_2\,(g_{11} - g_{33} \pm \sqrt{(g_{11} + g_{33})^2 - 4g_{13}^2}, \quad \lambda_2 = 0$$

The quantites λ_1 and λ_3 are functionally independent and take values within the limits $C \leqslant \lambda_1 \leqslant A$, $-A \leqslant \lambda_3 \leqslant -C$.

The eigenvectors $\mathbf{w}_i\ (i = 1, 2, 3)$ have in the basis $\mathbf{u}_{A,B,C}$ the coordinates

$$\left(\frac{x_i\alpha - y_i\beta + z_i\gamma}{A}, \frac{x_i\alpha' + y_i\beta' + z_i\gamma'}{B}, \frac{x_i\alpha'' + y_i\beta'' + z_i\gamma''}{C}\right)$$

where

$$x_{1,3} = \lambda_{1,3}(\sqrt{g_+} \pm \sqrt{g_-}), \quad y_{1,3} = \lambda_{1,3}(\sqrt{g_+} \mp \sqrt{g_-})$$
$$z_{1,3} = (g_{12} \mp g_{23})\sqrt{g_+} + (g_{23} \pm g_{12})\sqrt{g_-}$$
$$g_\pm = {}^1/_2(g_{11} + g_{33}) \pm g_{13}, \quad x_2 = y_2 = 0, \quad z_2 = 1$$

Since $V = a(\lambda_1 + \lambda_3)$, $U = -a\lambda_1\lambda_3$, the characteristic functions are $\Phi_{1,3} = a\lambda_{1,3}^2 + h\lambda_{1,3} - f$, $\Phi_2 = U - f$.

The regions where motion is possible $M_{fh} = \{\Phi_1 \geqslant 0\} \cap \{\Phi_3 \leqslant 0\}$ are bounded by the surfaces $\{\lambda_{1,3} = \mathrm{const}\}$ and are determined by the disposition of the roots of the quadratic trinomial $a\lambda^2 + h\lambda - f$ with respect to the intervals $[C, A]$ and $[-A, -C]$.

The critical points of the characteristic functions are defined by the equations

$$0 = d\Phi_{1,3} = (2a\lambda_{1,3} + h)\, d\lambda_{1,3}, \quad 0 = d\Phi_2 = dU$$

The conditions $dU = 0$, $d\lambda_{1,3} = 0$ provide the points $\mathbf{u}_{A,B,C} \parallel \mathbf{e}_{1,2,3}$, determined earlier, and the conditions $2a\lambda_{1,3} + h = 0$ define new sets of critical points that is, the surfaces $\{\lambda_{1,3} = \mathrm{const}\}$. To the critical zero level of the characteristic function $\Phi_1|_{fh}$ there corresponds the constant quantity λ_1 equal to the multiple root of the trinomial $a\lambda^2 + h\lambda$ in the interval $[C, A]$. This trinomial is positive on $[-A, -C]$, i.e. everywhere we have $\Phi_3|_{fh} > 0$, and hence M_{fh} is empty. On the critical zero level $\{\lambda_3 = \mathrm{const}\}$ of the characteristic function $\Phi_3|_{fh}$ we similarly have $\Phi_1|_{fh} > 0$, i.e. $M_{fh} = \{\lambda_3 = \mathrm{const}\}$.

By the second theorem in Sect.2 this surface M_{fh} consists of the trajectories of the problem on which the velocity $\mathbf{v} = \omega\mathbf{w}_3$, $\omega = \pm\sqrt{2(h - V)}\,|\mathbf{w}_3|^{-1}$.

Using the integral G, we obtain the function of positional variables $G(\omega\mathbf{w}_3)$ that are functionally independent of λ_3. Since on these trajectories $G(\omega\mathbf{w}_3) = \mathrm{const}$, $\lambda_3 = \mathrm{const}$, they are closed curves and the motions on them are periodic.

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NON-LINEAR ANALYSIS OF THE STABILITY OF THE LIBRATION POINTS OF A TRIAXIAL ELLIPSOID*

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The stability of libration points of a triaxial homogeneous gravitating ellipsoid rotating around one of its principal central axes of inertia is studied. The plane motion of a passive point of unit mass is considered. In parameter space a region of stability is constructed and, also, resonance sets for all the resonances investigated. A systematic analysis of the stability of a libration point is carried out, using respective theorems for the equilibrium positions of Hamiltonian systems with two degrees of freedom.

A qualitative investigation of the geometric structure of the stability region was carried out in /1, 2/.

If the ellipsoid is a figure of revolution around the central polar axis of inertia /1/, the relative equilibrium positions are not isolated and fill a circle in the equatorial plane. If, however, the equatorial semiaxes are different, the ellipsoid may have up to four isolated positions of relative equilibrium. The conditions of existence of libration points external to the ellipsoid in this problem and, also, the canonical equations of motion in the

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